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# Comparison of the Extended Linear Sigma Model and Chiral Perturbation Theory for Nucleon Properties

**M. Abu-Shady · M. Rashdan**

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**Abstract** We compare the extended linear sigma model and chiral perturbation theory in describing the static nucleon properties. A good description of some nucleon properties is obtained by the extended linear sigma model.

## 1 Introduction

One of the effective models in describing hadron properties is the linear sigma model [1], which serves as a good low energy effective theory in order for one to gain some insight into QCD. It has attracted much attention, especially in studies involving disoriented chiral condensates (DCCs) [2]. The model is very well suited for describing the physics of pions in studies of chiral symmetry. Fermions may be included in the model either as nucleons, if one is to study nucleon interactions, or as quarks. The mesonic part of the model consists of four scalar fields, one scalar isoscalar field which is called the sigma field and the usual three pion fields $\pi_0, \pi_\pm$ which form a pseudoscalar isovector. Some of the consequences of this model, however are known to be in conflict with observation. Notably, the isoscalar pion-nucleon $(\pi N)$ scattering length predicted more the model is larger than the experimental value by an order of magnitude; see Refs. [3, 4]. Rashdan et al. [5] introduced a new version of the linear sigma model with higher-order mesonic interactions, which we call the extended linear sigma model to improve nucleon properties.

Recently, chiral perturbation theory is the effective theory for low energy QCD. It is a Lagrangian theory written as a collection of derivatives of scalar field. The theory organizes the Lagrangians in powers of the momentum of the hadrons and performs ordinary field theoretical perturbation theory where the higher order Lagrangians provide counter terms which regularize the theory [6]. With these Lagrangians one can make predictions

M. Abu-Shady (✉)
Department of Mathematics, Faculty of Science, Menofia University, Shebin El-Kom, Egypt
e-mail: abu_shady_1999@yahoo.com

M. Rashdan
Department of Mathematics and Theoretical Physics, AEA, Cairo, Egypt

for meson-meson interaction and meson baryon interaction at lowest order, which reproduce the results obtained with current algebra techniques, yet in a more elegant, systematic and technically simpler way. The perturbative technique allows a fully quantized treatment of the pion field up to a given order in accuracy. Although formulated on the quark level, where confinement is put in phenomenologically, perturbative chiral quark models are formally close to chiral perturbation theory on the hadron level. As a further development of chiral quark models with a perturbative treatment of the pion cloud [7–12], the Tuebingen group extended the relativistic quark model for the study of the low-energy properties of the nucleon [13, 14]. Compared to the previous similar models of Refs. [7–11] this current approach contains several new features: (i) generalization of the phenomenological confining potential; (ii) SU(3) extension of chiral symmetry to include the kaon and eta-meson cloud contributions; (iii) consistent formulation of perturbation theory both on the quark and baryon level by use of renormalization techniques and by allowing to account for excited quark states in the meson loop diagrams; (iv) fulfillment of the constraints imposed by chiral symmetry (low-energy theorems), including the current quark mass expansion of the matrix elements (for details see Ref. [12]); (v) possible consistency with chiral perturbation theory as for example demonstrated [12] for the chiral expansion of the nucleon mass. This paper is organized as follows: In Sect. 2, we give a brief summary of the chiral perturbation theory and the extended linear sigma model. In Sect. 3, we compare the results between two models for describing nucleon properties.

## 2 Theoretical Description of the Chiral Perturbation Theory and Extended Linear Sigma Model

### 2.1 The Chiral Perturbation Theory

Recently, the perturbative chiral quark model (PCQM) play a very important role to calculate electromagnetic structure of nucleons [12]. This is model is based on an effective chiral Lagrangian describing the valence quarks of effective chiral Lagrangian describing the valence quarks of baryons as relativistic fermions moving in a self-consistent field (static potential, $V_{eff} = S(r) + \gamma^0 \mathbf{V}(r)$, with $r = |\mathbf{x}|$ [13–15]), which are supplemented by a cloud of Goldstone bosons $(\pi, k, \eta)$. Treating Goldstone fields as small fluctuations around the three-quark (3q) core we derive a linearized effective Lagrangian $l_{eff}$. The Lagrangian $l_{eff} = l_{inv}^{lin} + l_{xSB}$, derived in Ref. [16], includes the linear chiral invariant term

$$l_{inv}^{lin} = \bar{\Psi}(x)[i\gamma_\mu \partial^\mu - S(r) - \gamma^0 V(r)]\Psi(x) + \frac{1}{2}[\partial_\mu \hat{\Phi}(x)]^2 - \bar{\Psi}(x)S(r)i\gamma^5 \frac{\hat{\Phi}(x)}{F}\Psi(x) \quad (1)$$

and a term $l_{xSB}$ which explicitly breaks chiral symmetry,

$$l_{xSB} = -\bar{\Psi}(x)M\Psi(x) - \frac{B}{8}Tr\{\hat{\Phi}(x), \{\hat{\Phi}(x), M\}\}, \quad (2)$$

containing the mass terms for quarks and mesons [16]. The octet matrix $\Phi(x)$ of pseudoscalar mesons is defined as:

$$\frac{\Phi}{\sqrt{2}} = \sum_{i=1}^{8} \frac{\Phi_i \lambda_i}{\sqrt{2}} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2\eta}{\sqrt{6}} \end{pmatrix}. \quad (3)$$

Here $F = 88$ MeV [13–15] is the pion decay constant in the chiral limit, $M = dig\{\hat{m}, \hat{m}, m_s\}$ is the mass matrix of current quarks $B = -\frac{\langle 0|\bar{u}u|0\rangle}{F^2}$ is the low-energy constant which measures the vacuum expectation value of the scalar quark densities in the chiral limit [16]. We rely on the standard picture of chiral symmetry breaking [17, 18] and for the masses of pseudoscalar mesons we use the leading term in their chiral expansion (i.e., linear in the current quark mass):

$$M_\pi^2 = 2\hat{m}B, \qquad M_k^2 = (\hat{m} + m_s)B, \qquad M_\eta^2 = \frac{2}{3}(\hat{m} + 2m_s)B. \tag{4}$$

Meson masses obviously satisfy the Gell-Mann-Oakes-Rennr equation (4) and Gell-Mann Okubo relation $3M_\eta^2 + M_\pi^2 = 4M_k^2$. In the evaluation we use the following set of QCD parameters [19]: $\hat{m} = 7$ MeV, $\frac{m_s}{\hat{m}} = 25$, $B = \frac{M_\pi^2}{2\hat{m}} = 1.4$ GeV.

Furthermore, the linearized effective Lagrangian fulfills the partial conservation of axial vector current (PCAC) requirement, consistent with the Goldberger-Treiman relation.

### 2.2 The Extended Linear Sigma Model

The extended linear sigma model is described (for details; see Ref. [5]) and in the following, we give a brief summary.

The Lagrangian density of the extended linear sigma model which describes the interactions between quarks via the $\sigma$- and $\vec{\pi}$-mesons is written as Ref. [5]

$$L(r) = i\overline{\Psi}\gamma_\mu\partial^\mu\Psi + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\,.\,\partial^\mu\vec{\pi}) + g\overline{\Psi}(\sigma + i\gamma_5\vec{\tau}\,.\,\vec{\pi})\Psi - U_2(\sigma, \vec{\pi}), \tag{5}$$

with

$$U_2(\sigma, \vec{\pi}) = \frac{\lambda_1^2}{4}((\sigma^2 + \vec{\pi}^2)^2 - \nu_1^2)^2 + m_\pi^2 f_\pi \sigma,$$

where

$$\nu_1^2 = f_\pi^4 - \frac{m_\pi^2}{2\lambda_1^2 f_\pi^2}, \tag{6}$$

and

$$\lambda_1^2 = \frac{m_\sigma^2 - 3m_\pi^2}{8f_\pi^6}. \tag{7}$$

Now, we expand the extremum, with the shifted field defined as

$$\sigma = \sigma' - f_\pi, \tag{8}$$

inserting (8) into (5), we obtain:

$$\begin{aligned} L(r) = {} & i\overline{\Psi}\gamma_\mu\partial^\mu\Psi + \frac{1}{2}(\partial_\mu\sigma'\partial^\mu\sigma' + \partial_\mu\vec{\pi}.\partial^\mu\vec{\pi}) \\ & - g\overline{\Psi} f_\pi\Psi + g\overline{\Psi}\sigma'\Psi + ig\overline{\Psi}\gamma_5\vec{\tau}.\vec{\pi}\Psi - U_2(\sigma', \pi), \end{aligned} \tag{9}$$

with

$$U_2(\sigma', \vec{\pi}) = \frac{\lambda_1^2}{4}((\sigma' - f_\pi)^2 + \vec{\pi}^2 - \nu^2)^2 + m_\pi^2 f_\pi \sigma' - m_\pi^2 f_\pi^2. \tag{10}$$

The time-independent fields $\sigma'(r)$ and $\vec{\pi}(r)$ are to satisfy the Euler-Lagrange equations, and the quark wave function satisfies the Dirac eigenvalue equation. The meson field equations are written as:

$$\Box\sigma' = g\overline{\Psi}\Psi - 2\lambda_1^2(\sigma' - f_\pi)((\sigma' - f_\pi)^2 + \vec{\pi}^2) \times (((\sigma' - f_\pi)^2 + \vec{\pi}^2)^2 - \nu^2) - m_\pi^2 f_\pi, \quad (11)$$

$$\Box\vec{\pi} = \overline{\Psi}\gamma_5 \cdot \vec{\tau}\Psi - 2\lambda_1^2\vec{\pi}((\sigma' - f_\pi)^2 + \vec{\pi}^2)(((\sigma' - f_\pi)^2 + \vec{\pi}^2)^2 - \nu^2), \quad (12)$$

where $\vec{\tau}$ refers to Pauli isospin matrices, $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We used hedgehog ansatz in the pion field [3] where

$$\vec{\pi}(r) = \hat{r}\pi(r). \quad (13)$$

Now, the pion isospin and space angular momentum are correlated because the quark source terms are themselves correlated corresponding to $SU_{spin}(2)XSU_{isospin}(2)$ wave functions. This will be established using hedgehog ansatz, which breaks $\vec{I}$ symmetry and breaks $\vec{J}$ symmetry, but conserves the Grand spin $\vec{G}$

$$\vec{G} = \vec{J} + \vec{I}. \quad (14)$$

The Dirac equation for the quarks are

$$\frac{du}{dr} = -p(r)u + (E - m_q + S(r))w, \quad (15)$$

$$\frac{dw}{dr} = -(E - m_q + S(r))u + \left(\frac{2}{r} - p(r)\right)w, \quad (16)$$

where $S(r) = g\langle\sigma'\rangle$, $p(r) = g\langle\vec{\pi} \,.\, \hat{r}\rangle$, $E$ are the scalar potential, the pseudoscalar potential and the eigenvalue of the quarks spinor $\Psi$, respectively. Including the color degrees of freedom, one has $g\overline{\Psi}\Psi \to N_c g\overline{\Psi}\Psi$ where $N_c = 3$ colors, $g$ is the coupling constant. The Dirac wave functions $\Psi(r)$ and $\overline{\Psi}(r)$ are given by

$$\Psi(r) = \frac{1}{\sqrt{4\pi}}\begin{bmatrix} u(r) \\ iw(r) \end{bmatrix} \quad \text{and} \quad \overline{\Psi}(r) = \frac{1}{\sqrt{4\pi}}\begin{bmatrix} u(r) & iw(r) \end{bmatrix}, \quad (17)$$

and the sigma, pion and vector densities are given by

$$\rho_s = N_c g\overline{\Psi}\Psi = \frac{3g}{4\pi}(u^2 - w^2), \quad (18)$$

$$\rho_p = iN_c g\overline{\Psi}\gamma_5.\vec{\tau}\Psi = \frac{3}{4\pi}g(-2uw), \quad (19)$$

$$\rho_v = \frac{3g}{4\pi}(u^2 + w^2). \quad (20)$$

These equations are subject to the boundary conditions that asymptotically the fields approach their vacuum values,

$$\sigma(r) \sim -f_\pi \text{ MeV}, \qquad \pi(r) \sim 0 \quad \text{at } r \to \infty \quad (21)$$

(for details; see Ref. [5]).

## 3 Results and Discussion

The field equations (11–16) have been solved by iteration method as in Refs. [3, 5] for different values of quark and sigma masses.

Recently, the free pion mass is found to be play important role to give good description of nucleon properties. However, the nucleon magnetic moments is expressed as $\mu_N(m_\pi) = \frac{\mu_N^0}{1+\alpha m_\pi+\beta m_\pi^2}$; where, $\mu_N^0$, $\alpha$ and $\beta$ are fitted phenomenologically (for details see Ref. [20]). So, the decreasing of free pion mass lead to improve of magnetic moment of proton and $\sigma(\pi N)$ term as seen from Table 1. Also, we found quark and sigma masses effect on observables of nucleon as seen from Tables 2, 3.

Attractive feature of our model in comparison with experimental data and chiral perturbation theory. All observables are greatly modified. In particular, $\sigma(\pi N)$ term is corrected and closed with experimental data as seen from Table 4. This is mainly due to the increase higher-order mesonic contributions in an extended linear sigma model.

**Table 1** Values of magnetic moments of proton and neutron, $\sigma(\pi N)$, at $m_q = 390$ MeV, $m_\sigma = 1300$ MeV. All quantities in MeV

| $m_\pi$ (MeV) | 100 | 120 | 140 |
|---|---|---|---|
| $\sigma(\pi N)$ | 43 | 55 | 67 |
| Total moment proton $\mu_p(N)$ | 2.80 | 2.64 | 2.48 |
| Total moment neutron $\mu_n(N)$ | −2.20 | −2.04 | −1.89 |

**Table 2** Values of magnetic moments of proton and neutron, $\sigma(\pi N)$, at $m_q = 450$ MeV, $m_\pi = 140$ MeV. All quantities in MeV

| $m_\sigma$ (MeV) | 700 | 900 | 1100 |
|---|---|---|---|
| $\sigma(\pi N)$ | 119 | 96 | 90 |
| Total moment proton $\mu_p(N)$ | 2.70 | 2.69 | 2.75 |
| Total moment neutron $\mu_n(N)$ | −2.04 | −2.07 | −2.14 |

**Table 3** Values of magnetic moments of proton and neutron, $\sigma(\pi N)$, at $m_\pi = 140$ MeV, $m_\sigma = 1300$ MeV. All quantities in MeV

| $m_q$ (MeV) | 390 | 400 | 420 |
|---|---|---|---|
| $\sigma(\pi N)$ | 67 | 73 | 79 |
| Total moment proton $\mu_p(N)$ | 2.48 | 2.59 | 2.69 |
| Total moment neutron $\mu_n(N)$ | −1.89 | −1.99 | −2.10 |

**Table 4** Values of some observables calculated from the extended linear sigma model (ELSM) compared with chiral perturbation theory ($\chi PT$)

| Quantity | $\chi PT$ [12, 16] | ELSM [5] | Exp. [4, 12] |
|---|---|---|---|
| Total moment proton $\mu_p(N)$ | $2.62 \pm 0.02$ | 1.80 | 2.793 |
| Total moment neutron $\mu_n(N)$ | $-2.02 \pm 0.02$ | −2.20 | −1.913 |
| $\sigma(\pi N)$ | 54.7 | 43 | $35 \pm 10$ |

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